Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 227 (2008) 6532–6552
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

A study of moving mesh PDE methods for numerical simulation of blowup in reaction diffusion equations

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Received 23 January 2008; accepted 16 March 2008
Available online 28 March 2008

Abstract

A new concept called the dominance of equidistribution is introduced for analyzing moving mesh partial differential equations for numerical simulation of blowup in reaction diffusion equations. Theoretical and numerical results show that a moving mesh method works successfully when the employed moving mesh equation has the dominance of equidistribution. The property can be verified using dimensional analysis. In several aspects the current work generalizes previous work where a moving mesh equation is shown to have this dominance of equidistribution if it preserves the scaling invariance of the underlying physical partial differential equation and uses a small, constant value for τ (a parameter used for adjusting response time of the mesh movement to the change in the physical solution). Also, cases with both constant and variable τ are considered here.

MSC: 65N50; 65M50; 35K55; 35K57; 65N35

Keywords: Moving mesh; Blowup; Reaction diffusion equations; Mesh adaptation

1. Introduction

We are concerned with the numerical solution of reaction diffusion equations whose solutions become unbounded (or blowup) in a finite time. This type of partial differential equation (PDE) arises from mathematical idealizations of models describing combustion in chemicals or chemotaxis in cellular aggregates, the formation of shocks in the inviscid Burgers' equation, and the space-charge equations; e.g. see Pao [18]. Such a blowup in the solution often represents an important change in the properties of the model, such as the ignition of a heated gas mixture, and it is important that it is reproduced accurately by a numerical computation.

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1 This work was supported in part by the NSF under Grants DMS-0410545 and DMS-0712935.
2 This work was supported in part by NSERC (Canada) Grant A8781.

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2008.03.024

Since a blowup typically occurs on increasingly small length scales as well as time scales, it is essential to use an adaptive mesh in the numerical simulation. Two types of mesh adaptation have been commonly used, mesh refinement [4] and mesh movement [8]. With the former approach, mesh points are added as the length scale is getting smaller whereas in the latter approach a fixed number of mesh points are moved to resolve the increasingly small length scale.

In this paper we are interested in the moving mesh solution of blowup problems and focus particularly on the MMPDE (moving mesh PDE) moving mesh method developed in [13]. It has been shown in [8] that the key to the success of the method is to have MMPDEs preserve the scaling invariance of the underlying physical equation. This idea has since been used with success in most computations of blowup solutions that use the MMPDE method; e.g. see [7]. However, as we shall see in Sections 4 and 5, preserving the scaling invariance is neither sufficient nor necessary in general for MMPDEs to work satisfactorily, although it is sufficient for the particular approach considered in [8] where the parameter τ used in MMPDEs for adjusting the response time of mesh movement to the change in the physical solution is taken to be constant. Approaches with variable τ have also been used successfully by a number of researchers; e.g. see [12,19]. Thus, from both the theoretical and practical points of view there is a need for an in-depth study of the moving mesh method for blowup problems.

The objective of this paper is to present such a study. We are most concerned with conditions under which MMPDEs work satisfactorily. The tool we use is a new concept called *the dominance of equidistribution*: the terms representing the well known equidistribution principle [10,11] for mesh adaptation dominate other terms in the equation. We show that the solution of an MMPDE stays closely to the solution of the equidistribution principle when it has this property, implying that the dominance of equidistribution is sufficient for an MMPDE to work satisfactorily. Moreover, we show that the dominance of equidistribution can often be straightforwardly verified using dimensional analysis. A special case of the dominance of equidistribution is to have an MMPDE preserve the scaling invariance of the underlying physical PDE and to choose a small τ – the approach used in [8]. Furthermore, the concept applies to general situations, including those with constant and variable τ, and even in multi-dimensions.

It is worth mentioning some history of numerical simulation of blowup. The first works on the topic are Nakagawa [16] and Nakagawa and Ushijima [17] where finite difference and finite element schemes on a uniform mesh are employed and analyzed for blowup for PDE (1) with $p = 2$. A mesh refinement strategy is proposed by Berger and Kohn [4] and a moving mesh method is presented by Budd et al. [8] for the numerical solution of blowup problems. A survey is given by Bandle and Brunner [2]. Recent works include [1,5–7,9].

An outline of the paper is as follows. The MMPDE method is described in the next section for a classic problem with blowup solutions. The dimensional analysis for both the physical and mesh equations is presented in Section 3. The question of how to verify the dominance of equidistribution using dimensional analysis is also addressed in this section. Theoretical and numerical analyses of MMPDEs with constant and solution-dependent τ are given in Sections 4 and 5, respectively. Additional comments and conclusions are contained in the final section.

2. Moving mesh PDE method

We study the moving mesh method for a classic problem with a blowup solution:

$$u_t = u_{xx} + u^p, \quad p > 1 \tag{1}$$

subject to the boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0. \tag{3}$$

It is known that when the initial solution is sufficiently large, the solution of the initial-boundary value problem tends to infinity at a point, say $x^* \in (0,1)$, as $t \to T$ for some finite time $T > 0$, x^* and T are referred to as the blowup point and time, respectively. A more precise description of the blowup profile of the solution is given in the following theorem (see [3] and references therein):

Theorem 2.1. *Let* $\beta = \frac{1}{p-1}$. *If the initial solution is sufficiently large, then the solution to the initial-boundary value problem (1)–(3) satisfies*

$$\lim_{t\to T}(T-t)^{\beta}u(x^*+\mu[(T-t)(\alpha-\log(T-t))]^{1/2},t)=\beta^{\beta}\left[1+\frac{\mu^2}{4p\beta}\right]^{-\beta} \tag{4}$$

uniformly for all $|\mu| \leqslant C$ *for a given constant* $C > 0$, *where* α *is a constant depending only on the initial solution.*

The theorem shows that the blowup profile can best be shown in the so-called kernel coordinate $\mu = \mu(x,t)$, which is fixed as $t \to T$ and defined as

$$\mu = (x-x^*)[(T-t)(\alpha-\log(T-t))]^{-1/2}. \tag{5}$$

A moving mesh of a fixed number of points is used in the moving mesh solution of the blowup problem. The mesh can be suitably defined through a coordinate transformation, viz., a mesh of N points can be expressed as

$$x_j(t) = x(\xi_j,t) \quad \text{with } \xi_j = \frac{j-1}{N-1}, \quad j=1,\ldots,N, \tag{6}$$

where $x = x(\xi,t)$ is a coordinate transformation between the computational domain $\Omega_c = [0,1]$ and the physical domain $\Omega = [0,1]$ and satisfies the boundary conditions

$$x(0,t) = 0, \quad x(1,t) = 1. \tag{7}$$

Moreover, a discretization of PDE (1) on the moving mesh can readily be carried out using this coordinate transformation. Transforming (1) from the old coordinates (x,t) to the new ones (ξ,t), we have

$$\dot{u} - \frac{u_\xi}{x_\xi}\dot{x} = \frac{1}{x_\xi}\frac{\partial}{\partial\xi}\left(\frac{u_\xi}{x_\xi}\right) + u^p, \tag{8}$$

where $\dot{u}$ and $\dot{x}$ denote the time derivatives in the new coordinates, i.e.,

$$\dot{u} = \frac{\partial u}{\partial t}(x(\xi,t),t)\bigg|_{\xi\ \text{fixed}}, \quad \dot{x} = \frac{\partial x}{\partial t}(\xi,t)\bigg|_{\xi\ \text{fixed}}.$$

Numerical methods can be used to discretize (8) on uniform meshes ξ_j, $j=1,\ldots,N$. In our numerical tests, we use the high-order conservative collocation methods introduced in [14] and the codes (MOVCOL) developed based on the methods.

For the MMPDE method, the coordinate transformation is determined as the solution of an MMPDE. Three MMPDEs in [13], MMPDE4, MMPDE5, and MMPDE6, are considered in this paper. They read as

$$-\tau\frac{\partial}{\partial\xi}\left(M\frac{\partial\dot{x}}{\partial\xi}\right) = \frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right), \tag{9}$$

$$\tau\dot{x} = \frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right), \tag{10}$$

$$-\tau\frac{\partial^2\dot{x}}{\partial\xi^2} = \frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right), \tag{11}$$

where $M = M(x,t)$ is the monitor function depending on the physical solution and used for controlling mesh concentration and $\tau > 0$ is a parameter used for adjusting the response time of mesh movement to changes in M. Following [8], we consider the monitor function in the general form

$$M(x,t) = u^{\gamma}, \tag{12}$$

where $\gamma > 0$ is a parameter. In our computation (using the code MOVCOL), the coupled system consisting of (8) and one of the MMPDEs is integrated for the physical solution and the mesh simultaneously.

MMPDEs (9)–(11) are formulated by adding a mesh speed term to the equidistribution principle that takes the form

$$\frac{\partial}{\partial\xi}\left(M\frac{\partial x}{\partial\xi}\right) = 0, \tag{13}$$

subject to the boundary conditions (7). The goal of such a formulation is to form a mesh equation that is of parabolic type and has a solution staying closely to the solution of (13), at least for small τ. The parabolic nature of the MMPDEs provides a degree of smoothness in mesh movement and also makes the integration of the mesh equation easier. On the other hand, having their solutions stay closely to that of (13) warrants a level of mesh adaptivity. Unfortunately, this approximation cannot always be guaranteed in practice, especially in our current situation where the monitor function M depends on a physical solution that becomes unbounded as the blowup develops. Nevertheless, we show in Sections 4 and 5 that this is the case when an MMPDE has the dominance of equidistribution – the left-hand side term is very small or even diminishing as $t \to T$ compared to the right-hand side term that represents the equidistribution principle. The dominance of equidistribution of the MMPDEs is investigated in the next section using dimensional analysis.

MMPDE4 and MMPDE6 have been considered in [8] for mesh movement for the simulation of blowup. It is shown that, when τ is chosen as a small constant and the monitor function (12) is chosen such that the MMPDEs preserve the scaling invariance of the original PDE (1), the resulting coordinate transformation takes the form

$$x(\xi,t) = x^* + (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}} z(\xi,t), \tag{14}$$

with the property

$$z(\xi,t) = z_0(\xi) + \mathrm{O}(\tau) + \mathrm{o}(1), \tag{15}$$

where $\mathrm{o}(1)$ denotes terms tending to zero as $t \to T$ and $z_0(\xi)$ is a function associated with the equidistribution principle (13). A coordinate transformation in the form (14) with the property (15) is desirable for blowup problems because, in this way, the computational coordinate ξ is a function of the kernel coordinate μ (cf. (5)) and, from Theorem 2.1, the blowup profile of the solution can be properly shown in the coordinate ξ. As will be seen in the next section, the MMPDEs have the dominance of equidistribution under these conditions.

3. Dimensional analysis, scaling invariance, and dominance of equidistribution

In this section we study dimensional analysis and scaling invariance for the physical and moving mesh PDEs, tools that will be employed in the next sections for the analysis of the MMPDEs. We also investigate the dominance of equidistribution of MMPDEs using dimensional analysis.

We begin with the physical PDE (1). Denote the dimensions of variables u, t, and x by $[u]$, $[t]$, and $[x]$, respectively. Then, the dimensions of the terms u_t, u_{xx}, and u^p in the equation are given by

$$[u_t] = \frac{[u]}{[t]}, \quad [u_{xx}] = \frac{[u]}{[x]^2}, \quad [u^p] = [u]^p.$$

The fact that all terms in the physical PDE are dimensionally homogeneous implies

$$\frac{[u]}{[t]} = \frac{[u]}{[x]^2} = [u]^p.$$

This yields the dimension relations

$$[x] = [t]^{\frac{1}{2}}, \quad [u] = [t]^{-\frac{1}{p-1}} = [t]^{-\beta}. \tag{16}$$

Thus, if the dimension of t is changed by a factor $\lambda > 0$, the dimensions of x and u must change by factors $\lambda^{1/2}$ and $\lambda^{-\beta}$, respectively, to keep the physical equation dimensionally balanced. This suggests, and it is easy to verify, that the PDE (1) be invariant under the scaling transformation

$$\begin{cases} t \to \lambda t, \\ x \to \lambda^{\frac{1}{2}} x, \qquad \forall\, \lambda > 0. \\ u \to \lambda^{-\beta} u, \end{cases} \tag{17}$$

We now analyze the dimensions of the MMPDEs. We first notice that the computational domain can always be chosen as the unit interval, so the computational coordinate is dimensionless. Denote the dimension of τ by $[\tau]$. Then the dimension equation for MMPDE5 reads as

$$\frac{[\tau][x]}{[t]} = [M][x],$$

or after simplification,

$$\frac{[\tau]}{[t]} = [M]. \tag{18}$$

For the monitor function in the form (12), the equation becomes

$$\frac{[\tau]}{[t]} = [u]^{\gamma}, \tag{19}$$

or using (16),

$$[\tau][t]^{\beta\gamma-1} = 1. \tag{20}$$

This indicates that the magnitude of the left-hand side term of MMPDE5 in (10) is on the order $[\tau][t]^{\beta\gamma-1}$ compared to that of the right-hand side term. For the situation where τ is taken as constant, we have $[\tau] = 1$. In addition, for the underlying physical PDE (1) the time scale can be taken as $[t] = T - t$, which becomes increasingly small as $t \to T$. Then from (20) we can see that the left-hand side term is vanishing as $t \to T$ when $\beta\gamma > 1$. In this case, the equidistribution term dominates and thus MMPDE5 has the dominance of equidistribution. On the other hand, it is easy to see that this will not happen when $\beta\gamma < 1$. The critical case is $\beta\gamma = 1$. In this case, (20) is balanced and MMPDE5 is dimensionally homogeneous and invariant under the scaling transformation (17). The dimension homogeneity means that all terms in the equation are of the same order of magnitude. Recalling that the constant parameter τ is contained in the left-hand side term, we can make the MMPDE equidistribution dominant by choosing τ sufficiently small.

MMPDE6 has the same dimension equation and thus the same dominance of equidistribution as MMPDE5.

The dimension equation of MMPDE4 reads as

$$\frac{[\tau][x][M]}{[t]} = [M][x],$$

or after simplification,

$$\frac{[\tau]}{[t]} = 1. \tag{21}$$

When τ is taken as constant, i.e., $[\tau] = 1$, the equidistribution term (the right-hand side term) is always dominated by the left-hand side term. In this case, MMPDE4 never has the dominance of equidistribution.

The above results are summarized in Table 1. The situation with solution-dependent τ is discussed in Section 5 and the result is summarized in Table 3.

Table 1
Summary of theoretical results for MMPDEs with constant τ

MMPDE	$M = u^{\gamma}$	$\tau > 0$	EP dom.	Scaling Inv.	Theory	Tests
MMPDE5 or MMPDE6	$\beta\gamma > 1$	Any	Yes	No	(39), (41), (42)	Fig. 1
	$\beta\gamma = 1$	Large	No	Yes	Unsatisfactory	Fig. 2
		Small	Yes	Yes	(49) and (50)	Fig. 3
	$\beta\gamma < 1$	Any	No	No	Unsatisfactory	Fig. 4
MMPDE4	$\beta\gamma > 0$	Any	No	No	Unsatisfactory	Fig. 5

"EP dom." is the abbreviation of the dominance of equidistribution and "Scaling Inv." for scaling invariance.

4. Moving mesh PDEs with constant τ

In this section we study MMPDE4, MMPDE5, and MMPDE6 with τ being taken as constant and the monitor function chosen as in (12). We are most interested in whether or not the formal analysis of the previous section is adequate for describing when they work satisfactorily and have dominance of equidistribution. The satisfaction is judged by whether or not they generate a coordinate transformation of the form (14) with property

$$z(\xi, t) \to z_0(\xi) \qquad \text{as} \quad t \to T. \tag{22}$$

As mentioned in Section 2, the solution profile in the peak region of blowup can be properly resolved in the computational coordinate ξ when the coordinate transformation is of the form (14) with property (22). The approach we use here is similar to that of [8], i.e., the MMPDEs are solved analytically using the exact form (4) for the solution $u(x,t)$. This approach is not practical in general since the physical solution is unknown and sought by the computation. Nevertheless, the analysis determines the "optimal" mesh for our particular problem. Moreover, the analysis in [8] for the semi-discrete system of the physical PDE coupled with the mesh equation shows that its self-similar solutions for u and x are consistent with those obtained by the approach assuming the exact form (4). In this sense, the results we obtain here can be viewed as continuous limits of the semi-discrete approximations. Finally, it should be emphasized that the approach is for theoretical analysis only. In actual computation, the monitor function is calculated using the computed solution and the resulting mesh is thus truly adaptive. The numerical results obtained this way are used to verify our theoretical findings in this and the following sections.

4.1. MMPDE5

We first consider MMPDE5 (10). Expanding the derivative on the right-hand side gives

$$\tau \dot{x} = M \frac{\partial^2 x}{\partial \xi^2} + \frac{\partial M}{\partial \xi} \frac{\partial x}{\partial \xi}. \tag{23}$$

We seek a coordinate transformation in the form (14). By differentiating (14) with respect to t and ξ, we have

$$\dot{x} = \frac{1}{2}(T-t)^{-\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}\left[-1 + [\alpha - \log(T-t)]^{-1}\right]z + (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}\dot{z}, \tag{24}$$

$$x_\xi = (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}} z_\xi, \tag{25}$$

$$x_{\xi\xi} = (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}} z_{\xi\xi}. \tag{26}$$

Using the exact form (4) for the solution $u(x,t)$, from (14) and (12) we have

$$u(x,t) = (T-t)^{-\beta}\beta^{\beta}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta} + o((T-t)^{-\beta}), \tag{27}$$

$$M = (T-t)^{-\beta\gamma}\beta^{\beta\gamma}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma} + o((T-t)^{-\beta\gamma}), \tag{28}$$

$$M_\xi = -\frac{\gamma}{2p}(T-t)^{-\beta\gamma}\beta^{\beta\gamma}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma-1} z z_\xi + o((T-t)^{-\beta\gamma}). \tag{29}$$

Inserting these results into (23) yields

$$\frac{\tau}{2}(T-t)^{-\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}[-1 + [\alpha - \log(T-t)]^{-1}]z + \tau(T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}\dot{z}$$
$$= (T-t)^{\frac{1}{2}-\beta\gamma}\beta^{\beta\gamma}[\alpha - \log(T-t)]^{\frac{1}{2}}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma} z_{\xi\xi} - \frac{\gamma\beta^{\beta\gamma}}{2p}(T-t)^{\frac{1}{2}-\beta\gamma}[\alpha - \log(T-t)]^{\frac{1}{2}}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma-1} z(z_\xi)^2$$
$$+ o\left((T-t)^{\frac{1}{2}-\beta\gamma}[\alpha - \log(T-t)]^{\frac{1}{2}}\right),$$

or after simplification,

$$\frac{\tau}{2}(T-t)^{\beta\gamma-1}[-1+[\alpha-\log(T-t)]^{-1}]z+\tau(T-t)^{\beta\gamma}\dot{z}$$
$$=\beta^{\beta\gamma}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma}z_{\xi\xi}-\frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma-1}z(z_\xi)^2+\mathrm{o}(1), \tag{30}$$

where $\mathrm{o}(1)$ denotes the terms tending to zero as $t\to T$. Thus, MMPDE5 reduces to (30), using (4), (12), and (14). In the following we study (30) in three separate cases: $\beta\gamma>1$, $\beta\gamma=1$, and $\beta\gamma<1$.

Case 1. $\beta\gamma>1$. In this case, MMPDE5 has the dominance of equidistribution; see Table 1. The property can also be checked by looking at the power of the factor $T-t$ in (30). It is not difficult to see that (30) permits a formal expansion for $z=z(\xi,t)$ as

$$z(\xi,t)=z_0(\xi)+\mathrm{o}(1), \tag{31}$$

where $z_0(\xi)$ satisfies the ordinary differential equation

$$\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}=\frac{\gamma}{2p}\frac{z_0}{1+\frac{z_0^2}{4p\beta}}\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2. \tag{32}$$

The above equation can be obtained directly from the equidistribution principle (13) using the same procedure, and therefore $z_0(\xi)$ can be considered as an approximation associated with equidistribution. Then (31) implies that the solution of the MMPDE5 stays closely to that of (13) as $t\to T$.

The boundary conditions for $z_0(\xi)$ are obtained as follows. From Theorem 2.1 and the form of the coordinate transformation, (14), we observe that for any given constant $C>0$, the mesh points $(x(\xi,t),t)$ with $|z(\xi,t)|,|\dot{z}(\xi,t)|\leqslant C$ will eventually lie in the blowup region of the solution as t approaches T. This, combined with expansion (31) and the fact that (14) is valid only in the blowup region, suggests

$$z_0(\xi^L)=z_0^L,\quad z_0(\xi^R)=z_0^R, \tag{33}$$

where ξ^L, ξ^R, z_0^L, and z_0^R are bounded constants with $-\infty<z_0^L,z_0^R<\infty$ and $0<\xi^L<\xi^R<1$. In theory these constants can be determined by matching the coordinate transformation from the inside and outside of the blowup peak region – e.g. see Kevorkian and Cole [15] for matching techniques. Unfortunately, this is a difficult task, if not impossible, for the current situation since analytic expressions for the physical solution and thus the coordinate transformation are unavailable in regions where $|x-x^*|$ is not small. Nevertheless, they can be estimated qualitatively. Indeed, since the equidistribution term in (10) is dominant as $t\to T$, the solution of the MMPDE closely satisfies the equidistribution relation (13), so

$$M(x,t)\frac{\partial x}{\partial\xi}=\sigma, \tag{34}$$

where $\sigma=\int_0^1M(x,t)\mathrm{d}x$. As a result, more mesh points are concentrated in the regions where M is larger and fewer points in the regions where M is smaller. For the monitor function (12), M is much larger in the peak region of blowup than the rest of the domain and thus most mesh points are concentrated in that region. This implies that the peak region spans almost the entire ξ domain $[0,1]$ since a uniform mesh in ξ is used in the computation (cf. (6)). Hence, ξ^L is close to zero and ξ^R close to 1, i.e.,

$$\xi^L\approx0,\quad\xi^R\approx1. \tag{35}$$

On the other hand, the boundary conditions for the coordinate transformation, $x(0,t)=0$ and $x(1,t)=1$, imply that the points $(x(0,t),t)$ and $(x(1,t),t)$ should stay outside of the peak. In form (14), they must correspond to the limits of large $|z|$: $z(0,t)=-\infty$ and $z(1,t)=\infty$. From (31) and (35) z_0^L and z_0^R, although bounded, should be very large, viz.,

$$z_0^L\approx-\infty,\quad z_0^R\approx\infty. \tag{36}$$

The boundary value problem for (32) and (33) can be solved by viewing z_0 as an independent variable. Letting

$$v = \frac{\mathrm{d}z_0}{\mathrm{d}\xi},$$

by the chain rule,

$$\frac{\mathrm{d}^2 z_0}{\mathrm{d}\xi^2} = \frac{\mathrm{d}v}{\mathrm{d}z_0}\frac{\mathrm{d}z_0}{\mathrm{d}\xi} = \frac{\mathrm{d}v}{\mathrm{d}z_0}v.$$

Eq. (32) can then be rewritten as

$$\frac{\mathrm{d}v}{\mathrm{d}z_0} = \frac{\gamma}{2p}\frac{z_0}{1+\frac{z_0^2}{4p\beta}}v,$$

whose solution takes the form

$$v = C\left(1+\frac{z_0^2}{4p\beta}\right)^{\beta\gamma}.$$

This gives

$$\frac{\mathrm{d}z_0}{\mathrm{d}\xi} = C\left(1+\frac{z_0^2}{4p\beta}\right)^{\beta\gamma}$$

or

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0} = C^{-1}\left(1+\frac{z_0^2}{4p\beta}\right)^{-\beta\gamma}. \tag{37}$$

Integrating this about z_0 from z_0^L to z_0^R and using the boundary conditions (33) yields

$$\frac{\xi-\xi^L}{\xi^R-\xi^L} = \frac{\int_{z_0^L}^{z_0}\left(1+\frac{s^2}{4p\beta}\right)^{-\beta\gamma}\mathrm{d}s}{\int_{z_0^L}^{z_0^R}\left(1+\frac{s^2}{4p\beta}\right)^{-\beta\gamma}\mathrm{d}s},$$

or

$$\frac{\xi-\xi^L}{\xi^R-\xi^L} = \frac{\frac{z_0}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{z_0^2}{4p\beta}\right) - \frac{z_0^L}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{(z_0^L)^2}{4p\beta}\right)}{\frac{z_0^R}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{(z_0^R)^2}{4p\beta}\right) - \frac{z_0^L}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{(z_0^L)^2}{4p\beta}\right)}. \tag{38}$$

Here, $F(a,b;c;z)$ is the Gauss hypergeometric function with scalar parameters a, b, c, and z, having properties

$$\int_0^z\left(1+\frac{s^2}{4p\beta}\right)^{-\beta\gamma}\mathrm{d}s = zF\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{z^2}{4p\beta}\right),$$

$$\lim_{z\to\infty}\frac{z}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{z^2}{4p\beta}\right) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\beta\gamma-\frac{1}{2})}{\Gamma(\beta\gamma)},$$

where Γ is the Gamma function. Recall that the boundary conditions for z_0 have the approximation properties (36) and (35). For this approximation, the solution (38) can be written as

$$\frac{z_0}{\sqrt{4p\beta}}F\left(\frac{1}{2},\beta\gamma;\frac{3}{2};-\frac{z_0^2}{4p\beta}\right) \approx \frac{\sqrt{\pi}\Gamma(\beta\gamma-\frac{1}{2})}{\Gamma(\beta\gamma)}\left(\xi-\frac{1}{2}\right). \tag{39}$$

We now consider the special case $\beta\gamma = \frac{3}{2}$. For this case, we have

$$F\left(\frac{1}{2},\frac{3}{2};\frac{3}{2};-\frac{z_0^2}{4p\beta}\right) = \frac{1}{\sqrt{1+\frac{z_0^2}{4p\beta}}} \quad \text{and} \quad \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

From (39) we get

$$\frac{z_0}{\sqrt{1+\frac{z_0^2}{4p\beta}}} \approx 2\sqrt{4p\beta}\left(\xi - \frac{1}{2}\right),$$

or

$$z_0(\xi) \approx \frac{4\sqrt{p\beta}\left(\xi - \frac{1}{2}\right)}{\sqrt{1 - 4\left(\xi - \frac{1}{2}\right)^2}}. \tag{40}$$

From (14) we obtain the coordinate transformation in the peak region of blowup as

$$x(\xi, t) = x^* + (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}\left(\frac{4\sqrt{p\beta}(\xi - \frac{1}{2})}{\sqrt{1-4\left(\xi - \frac{1}{2}\right)^2}} + \mathrm{o}(1)\right). \tag{41}$$

The solution in the region can be expressed in terms of the computational coordinate as

$$u(x(\xi,t),t) = (T-t)^{-\beta}\beta^{\beta}\left[\left(1 - 4(\xi - \frac{1}{2})^2\right)^{\beta} + \mathrm{o}(1)\right]. \tag{42}$$

Case 2. $\beta\gamma = 1$. As we have seen in Section 3, this is a critical case. MMPDE5 is scaling invariant and all terms in the equation are dimensionally homogeneous. Moreover, the MMPDE has the dominance of equidistribution when τ is sufficiently small.

From (30) we see that $z(\xi, t)$ can be expanded into the form (31), with z_0 satisfying

$$\frac{\mathrm{d}^2 z_0}{\mathrm{d}\xi^2} = \frac{\gamma}{2p}\left(1 + \frac{z_0^2}{4p\beta}\right)^{-1} z_0 \left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2 - \frac{\tau}{2\beta}\left(1 + \frac{z_0^2}{4p\beta}\right) z_0. \tag{43}$$

Letting

$$v = \frac{\mathrm{d}z_0}{\mathrm{d}\xi},$$

(43) becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}z_0} = \frac{\gamma}{2p}\left(1 + \frac{z_0^2}{4p\beta}\right)^{-1} z_0 v^2 - \frac{\tau}{2\beta}\left(1 + \frac{z_0^2}{4p\beta}\right) z_0$$

or

$$\frac{\mathrm{d}(v^2)}{\mathrm{d}(z_0^2)} = \frac{\gamma}{2p}\left(1 + \frac{z_0^2}{4p\beta}\right)^{-1} v^2 - \frac{\tau}{2\beta}\left(1 + \frac{z_0^2}{4p\beta}\right).$$

This is a linear equation for v^2 by viewing z_0^2 as independent variable. Its solution can be found as

$$\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2 = \left[-2\tau p \ln\left(1 + \frac{z_0^2}{4p\beta}\right) + C\right]\left(1 + \frac{z_0^2}{4p\beta}\right)^2, \tag{44}$$

where C is an integration constant.

For large τ, we consider the ODE (44) with boundary conditions (33). Notice that the constant C depends on the boundary conditions. Since the approximation (36) is valid, the constant C must be positive and large to keep the right-hand side term of ODE (44) positive. Therefore, the ODE (44) with boundary conditions (36) has no solutions. To keep this from happening, ξ^L and ξ^R must be far away from the points 0 and 1, respectively. This also means few points are distributed in the blowup peak region, which conflicts with the role of mesh adaptation. Thus, MMPDE5 works unsatisfactorily for the case $\beta\gamma = 1$ with large τ.

Next, we consider the situation when τ is sufficiently small. From (14), we see that

$$z_\xi = x_\xi[(T-t)(\alpha - \log(T-t))]^{-1/2}.$$

It follows that $z_\xi > 0$ for $0 < T - t \leqslant 1$ since the moving mesh density $x_\xi > 0$. From the asymptotic relation between $z(\xi, t)$ and $z_0(\xi)$ (see (22)), $z_0(\xi)$ is monotonically increasing. The boundary conditions (33) then imply

$$z_0^L \leqslant z_0 \leqslant z_0^R.$$

Thus, when τ is chosen sufficiently small such that

$$2\tau p \ln\left(1 + \frac{(z_0^L)^2}{4p\beta}\right) \ll 1, \quad 2\tau p \ln\left(1 + \frac{(z_0^R)^2}{4p\beta}\right) \ll 1,$$

(44) can be written as

$$\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2 = [C + \mathrm{O}(\tau)]\left(1 + \frac{z_0^2}{4p\beta}\right)^2, \tag{45}$$

which is equivalent to

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0} = [C + \mathrm{O}(\tau)]^{-\frac{1}{2}}\left(1 + \frac{z_0^2}{4p\beta}\right)^{-1}. \tag{46}$$

Integrating this equation about z_0 from z_0^L to z_0^R and applying the boundary conditions (33) give

$$\frac{\xi - \xi^L}{\xi^R - \xi^L} = \frac{\arctan\left(\frac{z_0}{\sqrt{4p\beta}}\right) - \arctan\left(\frac{z_0^L}{\sqrt{4p\beta}}\right)}{\arctan\left(\frac{z_0^R}{\sqrt{4p\beta}}\right) - \arctan\left(\frac{z_0^L}{\sqrt{4p\beta}}\right)} + \mathrm{O}(\tau). \tag{47}$$

Using the approximations (36) and (35), we obtain

$$\xi \approx \frac{\arctan\left(\frac{z_0}{\sqrt{4p\beta}}\right) + \frac{\pi}{2}}{\pi} + \mathrm{O}(\tau)$$

or

$$z_0(\xi) \approx \sqrt{4p\beta}\tan\left(\pi\left(\xi - \frac{1}{2}\right)\right) + \mathrm{O}(\tau). \tag{48}$$

Then the coordinate transformation and the physical solution in the peak region of blowup satisfy

$$x(\xi, t) = x^* + (T-t)^{\frac{1}{2}}[\alpha - \log(T-t)]^{\frac{1}{2}}\left(\sqrt{4p\beta}\tan\left(\pi\left(\xi - \frac{1}{2}\right)\right) + \mathrm{O}(\tau) + \mathrm{o}(1)\right), \tag{49}$$

$$u(x(\xi, t), t) = (T-t)^{-\beta}\beta^\beta\left[\cos^{2\beta}\left(\pi\left(\xi - \frac{1}{2}\right)\right) + \mathrm{O}(\tau) + \mathrm{o}(1)\right]. \tag{50}$$

It is emphasized that (49) and (50) are valid only for small τ. They have been obtained in [8] for MMPDE6 using a scaling invariance argument. As will be seen in the next subsection, MMPDE5 and MMPDE6 lead to almost the same coordinate transformation when τ is small.

Case 3. $0 < \beta\gamma < 1$. For this case, the MMPDE does not have the dominance of equidistribution. This can also be seen by checking the power of the factor $T - t$ in (30). Moreover, the solution $z(\xi, t)$ to (30) cannot be expanded into

$$z(\xi, t) = z_0(\xi) + \mathrm{O}(\tau) + \mathrm{o}(1) \quad \text{or} \quad z(\xi, t) = z_0(\xi) + \mathrm{o}(1).$$

Thus, we cannot expect that the blowup peak of the solution can be properly resolved on an adaptive moving mesh determined by MMPDE5 with the current choice of the monitor function.

4.2. MMPDE6

The MMPDE6 has the form (11). The main difference between MMPDE5 and MMPDE6 lies in the left-hand term. Using (14) and (12) we have

$$-\frac{\tau}{2}(T-t)^{\beta\gamma-1}[-1+[\alpha-\log(T-t)]^{-1}]z_{\xi\xi}-\tau(T-t)^{\beta\gamma}\dot{z}_{\xi\xi}$$
$$=\beta^{\beta\gamma}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma}z_{\xi\xi}-\frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma-1}z(z_\xi)^2+\mathrm{o}(1). \tag{51}$$

This equation is similar to (30) obtained for MMPDE5 except that z and $\dot{z}$ on the left-hand side are replaced here by $z_{\xi\xi}$ and $\dot{z}_{\xi\xi}$, respectively. Since these changes do not alter the power of the factor $(T-t)$, the analysis given in the previous subsection for MMPDE5 for cases $\beta\gamma>1$ and $\beta\gamma<1$ also works for MMPDE6. In particular, for the case $\beta\gamma=\frac{3}{2}(>1)$ the coordinate transformation and the physical solution in the peak region are given in (41) and (42), respectively.

On the other hand, the situation for the critical case $\beta\gamma=1$ is different. Instead of (43), we now have

$$\left[1-\frac{\tau}{2\beta}\left(1+\frac{z_0^2}{4p\beta}\right)\right]\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}=\frac{\gamma}{2p}\left(1+\frac{z_0^2}{4p\beta}\right)^{-1}z_0\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2. \tag{52}$$

Letting

$$v=\frac{\mathrm{d}z_0}{\mathrm{d}\xi},$$

we have

$$\frac{\mathrm{d}v}{v}=\left[1+\frac{z_0^2}{4p\beta}\right]^{-1}\left[1-\frac{\tau}{2\beta}\left(1+\frac{z_0^2}{4p\beta}\right)\right]^{-1}\mathrm{d}\left(1+\frac{z_0^2}{4p\beta}\right).$$

Its solution can be found as

$$v=C\left|\frac{1+\frac{z_0^2}{4p\beta}}{1-\frac{\tau}{2\beta}\left(1+\frac{z_0^2}{4p\beta}\right)}\right|,$$

or

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0}=C^{-1}\left|\left(1+\frac{z_0^2}{4p\beta}\right)^{-1}-\frac{\tau}{2\beta}\right|. \tag{53}$$

We first consider the case with $\tau/(2\beta)>1$. Then the above equation becomes

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0}=C^{-1}\left[\frac{\tau}{2\beta}-\left(1+\frac{z_0^2}{4p\beta}\right)^{-1}\right].$$

Integrating this equation and using the boundary condition (33) yields

$$\frac{\xi-\xi^L}{\xi^R-\xi^L}=\frac{\frac{\tau}{2\beta}(z_0-z_0^L)-\sqrt{4p\beta}\left(\arctan\frac{z_0}{\sqrt{4p\beta}}-\arctan\frac{z_0^L}{\sqrt{4p\beta}}\right)}{\frac{\tau}{2\beta}(z_0^R-z_0^L)-\sqrt{4p\beta}\left(\arctan\frac{z_0^R}{\sqrt{4p\beta}}-\arctan\frac{z_0^L}{\sqrt{4p\beta}}\right)},$$

or

$$\frac{\tau}{2\beta}z_0-\sqrt{4p\beta}\arctan\frac{z_0}{\sqrt{4p\beta}}=\frac{\tau}{4\beta}(z_0^R+z_0^L)-\frac{\sqrt{4p\beta}}{2}\left(\arctan\frac{z_0^R}{\sqrt{4p\beta}}+\arctan\frac{z_0^L}{\sqrt{4p\beta}}\right)$$
$$+\frac{\xi-(\xi^L+\xi^R)/2}{\xi^R-\xi^L}\left[\frac{\tau}{2\beta}(z_0^R-z_0^L)-\sqrt{4p\beta}\left(\arctan\frac{z_0^R}{\sqrt{4p\beta}}-\arctan\frac{z_0^L}{\sqrt{4p\beta}}\right)\right]. \tag{54}$$

From (36) we know that $(\tau/(2\beta))(z_0^R - z_0^L)$ will be very large, and then there is no expression for z_0 when (36) applies. In other words, $z_0(\xi)$ will become very large (and therefore the corresponding physical mesh points stay outside of the peak region) even when ξ is not close to ξ^L or ξ^R but some distance away from the midpoint $(\xi^L + \xi^R)/2$. This conflicts with the goal of mesh adaptation that a large proportion of mesh points are concentrated in the peak region. In this sense, MMPDE6 is unsatisfactory in generating a proper adaptive mesh for the current case $\beta\gamma = 1$ for large τ. Note that the analysis can be straightforwardly extended to other values of τ that are not sufficiently small.

When τ is sufficiently small, on the other hand, (53) can be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0} = C^{-1}\left[\left(1 + \frac{z_0^2}{4p\beta}\right)^{-1} + \mathrm{O}(\tau)\right].$$

This equation is similar to (46) and the coordinate transformation and the physical solution in the peak region of blowup take the forms as in (49) and (50).

4.3. MMPDE4

For MMPDE4 (9) we have

$$\begin{aligned}
&-\tau\beta^{\beta\gamma}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma}\left[\frac{1}{2}(T-t)^{-1}[-1 + [\alpha - \log(T-t)]^{-1}]z_{\xi\xi} + \dot{z}_{\xi\xi}\right] \\
&\quad + \frac{\tau\gamma\beta^{\beta\gamma}}{2p}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma-1} z z_\xi \left[\frac{1}{2}(T-t)^{-1}[-1 + [\alpha - \log(T-t)]^{-1}]z_\xi + \dot{z}_\xi\right] \\
&\quad = \beta^{\beta\gamma}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma} z_{\xi\xi} - \frac{\gamma\beta^{\beta\gamma}}{2p}\left[1 + \frac{z^2}{4p\beta}\right]^{-\beta\gamma-1} z(z_\xi)^2 + \mathrm{o}(1).
\end{aligned} \tag{55}$$

It is not difficult to verify that the right-hand side term that corresponds to the equidistribution term in (9) is dominated by the left-hand side term as $t \to T$, and the solution $z(\xi, t)$ for (55) cannot be expanded in the form (31). As a result, this MMPDE, together with monitor function (12), do not lead to a coordinate transformation in the form (14) with $z(\xi, t)$ satisfying the property (31).

For convenience we summarize in Table 1 the theoretical results obtained in this section. We can see that an MMPDE works satisfactorily when the equidistribution term dominates as $t \to T$. This is not surprising since in this way the solution of the MMPDE stays close to that satisfying the equidistribution principle. One can also see from the table that preservation of scaling invariance is neither sufficient nor necessary for an MMPDE to work satisfactorily.

4.4. Numerical examples

We now present numerical results to verify the theoretical findings in this section. These results are obtained with MOVCOL developed in [14]. The initial solution is taken as

$$u_0(x) = 20\sin(\pi x). \tag{56}$$

The number of mesh points is chosen as 41 for all computations presented in this paper.

For the purpose of verification of the theoretical results, we plot the scaled solution profile, $u/\|u\|_\infty$, as function of ξ for several values of $\|u\|_\infty$. Note that different values of $\|u\|_\infty$ correspond to different instants in time.

We also plot $|x_i - x^*|$ against $\|u\|_\infty$ in logarithmic scale. To explain these functions, we take (41) and (42) as an example. Then we have

$$\|u\|_\infty \approx (T-t)^{-\beta}\beta^\beta \quad \text{or} \quad (T-t) \approx \beta\|u\|_\infty^{-\frac{1}{\beta}}.$$

It follows that, as $t \to T$,

$$\frac{u}{\|u\|_\infty} \to \left(1 - 4\left(\xi - \frac{1}{2}\right)^2\right)^\beta \tag{57}$$

and

$$\log|x_i - x^*| \to -\frac{1}{2\beta}\log\|u\|_\infty + \log\frac{4\sqrt{p\beta}(\xi_i - \frac{1}{2})}{\sqrt{1-4(\xi_i-\frac{1}{2})^2}} + \frac{1}{2}\log\beta + \frac{1}{2}\log\left[\alpha + \frac{1}{\beta}\log\|u\|_\infty - \log\beta\right]$$
$$\sim -\frac{1}{2\beta}\log\|u\|_\infty + c_i, \tag{58}$$

where c_i is a constant depending on ξ_i. Thus, when MMPDE5 works satisfactorily, the computed solution $u/\|u\|_\infty$ converges to a steady-state profile $\left(1 - 4(\xi - \frac{1}{2})^2\right)^\beta$ while $\log|x_i - x^*|$ is becoming linear in $\log\|u\|_\infty$ for most mesh points in the limit $t \to T$.

Numerical results are shown in Table 2 and Figs. 1–5. It is not difficult to see that they are consistent with the theoretical predictions. In particular, one can see that for the unsatisfactory situations shown in Figs. 2, 4, and 5, fewer and fewer mesh points are concentrated in the peak region of blowup (which is getting narrower as $t \to T$) and the solution $u/\|u\|_\infty$ is becoming more like a delta function as $t \to T$. On the other hand, for

Table 2
Summary of numerical results for MMPDEs with constant τ

MMPDE	τ	$M = u^\gamma$	p	Figs.	Satisfactory	EP dom.	Scaling Inv.
MMPDE5	10^{-5}	$\beta\gamma = 1.5$	3	Not shown	Yes	Yes	No
	10^2	$\beta\gamma = 1.5$	3	1	Yes	Yes	No
	10^{-5}	$\beta\gamma = 1$	2	Not shown	Yes	Yes	Yes
	10^2	$\beta\gamma = 1$	2	2	No	No	Yes
	10^{-5}	$\beta\gamma = 1$	3	3	Yes	Yes	Yes
	10^{-2}	$\beta\gamma = 2/3$	3	4	No	No	No
MMPDE6	10^{-5}	$\beta\gamma = 1.5$	3	Not shown	Yes	Yes	No
	10^2	$\beta\gamma = 1.5$	3	Not shown	Yes	Yes	No
MMPDE4	10^{-5}	$\beta\gamma = 1.5$	3	5	No	No	No

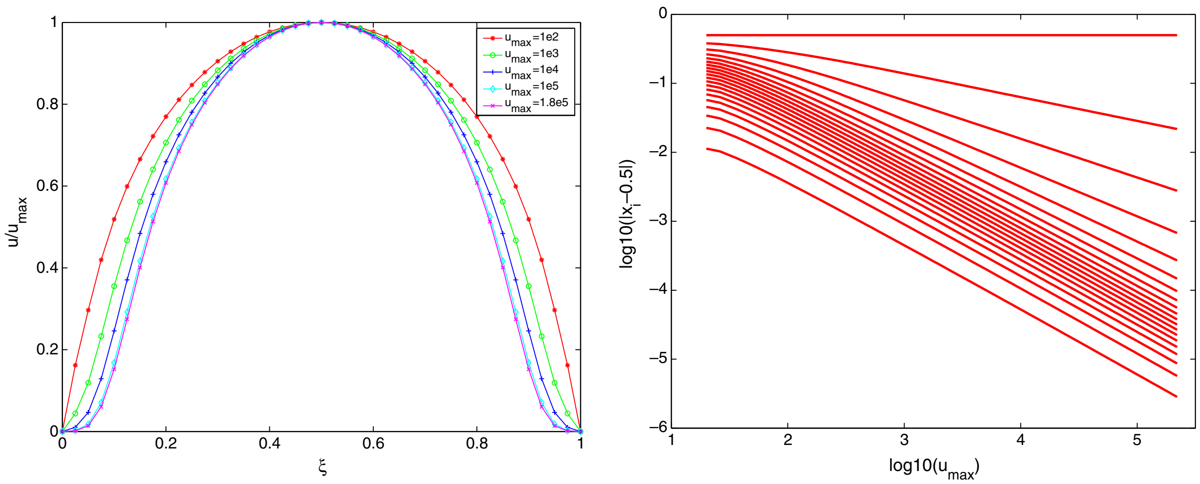

Fig. 1. MMPDE5, $M = u^{1.5(p-1)}$, $p = 3$, $\tau = 10^2$, $\beta\gamma = 1.5$.

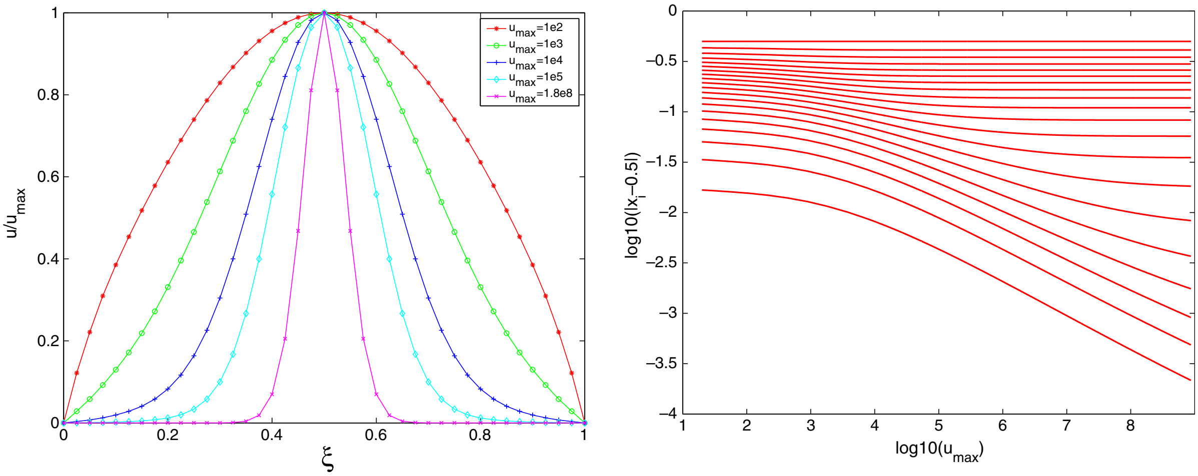

Fig. 2. MMPDE5, $M = u^{p-1}$, $p = 2$, $\tau = 10^2$, $\beta\gamma = 1$.

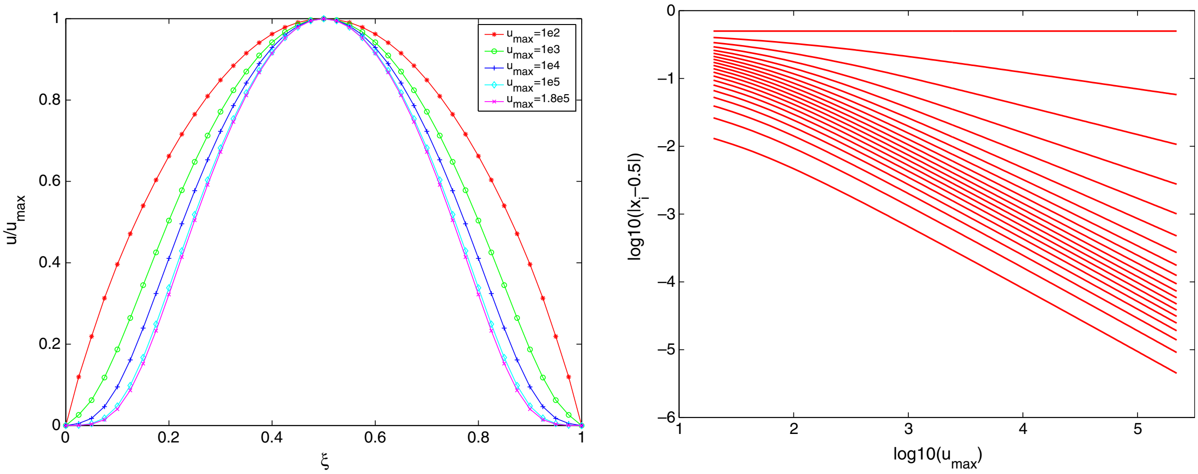

Fig. 3. MMPDE5, $M = u^{p-1}$, $p = 3$, $\tau = 10^{-5}$, $\beta\gamma = 1$.

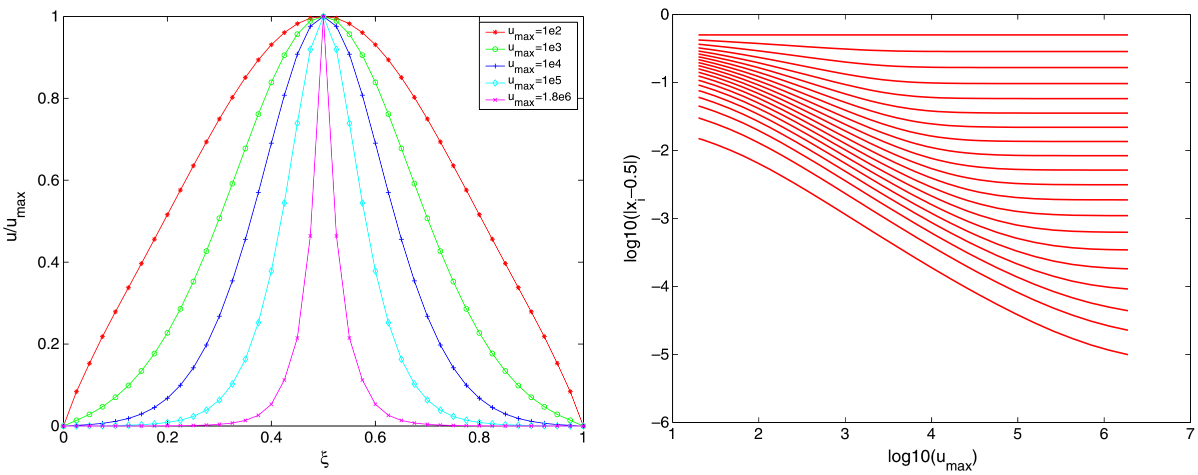

Fig. 4. MMPDE5, $M = u^{2(p-1)/3}$, $p = 3$, $\tau = 10^{-2}$, $\beta\gamma = 2/3$.

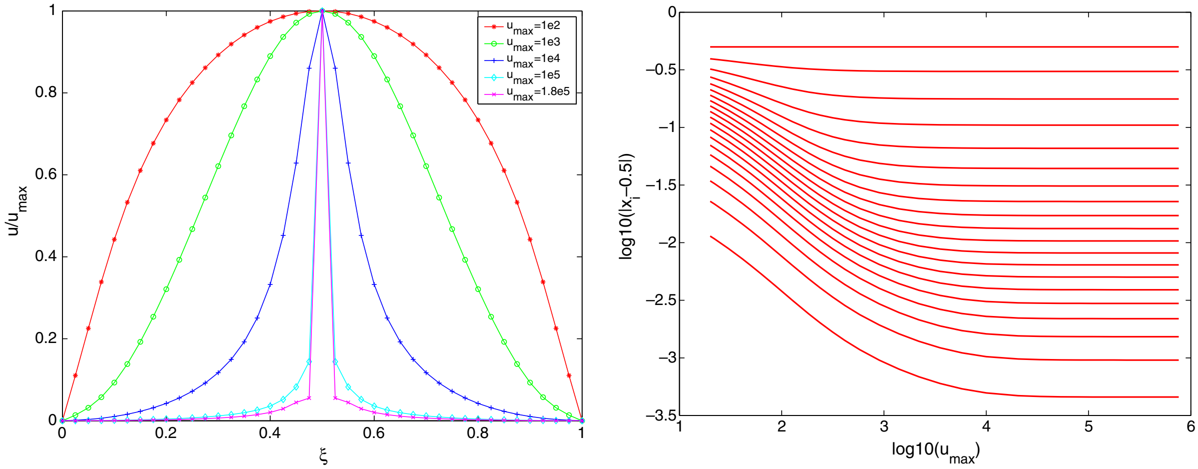

Fig. 5. MMPDE4, $M = u^{1.5(p-1)}$, $p = 3$, $\tau = 10^{-5}$, $\beta\gamma = 1.5$.

satisfactory situations most of mesh points are retained in the peak region of blowup while the solution $u/\|u\|_\infty$ is converging to a limit profile. More importantly, the numerical results show that MMPDEs work satisfactorily when they have the dominance of equidistribution.

5. Moving mesh PDEs with variable τ

We now study the MMPDEs with τ taken as a time-dependent function depending on the solutions. Here we use the strategy in [12] with which τ is defined such that the underlying MMPDE is dimensionally homogeneous. Again, we choose M in the general form (12) with $\gamma > 0$.

5.1. MMPDE5

For the dimension Eqs. (20) and (16) we have

$$[\tau] = [t]^{1-\beta\gamma} = [u]^{\gamma-\frac{1}{\beta}}.$$

To make this equation balanced, we choose τ as

$$\tau = \kappa u^{\gamma-\frac{1}{\beta}}, \tag{59}$$

where $\kappa > 0$ is a dimensionless constant. With this choice of τ, MMPDE5 is dimensionally homogeneous and all the terms contained in the equation are of the same order of magnitude. MMPDE5 can be made to have the dominance of equidistribution if κ is chosen sufficiently small. This argument applies to MMPDE4 and MMPDE6 with corresponding choices of τ.

Note that if $\beta\gamma = 1$, then $\tau = \kappa$ is constant. The case has been discussed in Section 4.1. So we assume that $\beta\gamma \neq 1$ in this subsection. With the choice of τ (59) and the form of the coordinate transformation (14), MMPDE5 can be simplified to

$$\kappa\beta^{\beta\gamma-1}\left[1+\frac{z^2}{4p\beta}\right]^{1-\beta\gamma}\left[\frac{1}{2}[-1+[\alpha-\log(T-t)]^{-1}]z+(T-t)\dot{z}\right]$$
$$=\beta^{\beta\gamma}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma}z_{\xi\xi}-\frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma-1}z(z_\xi)^2+o(1). \tag{60}$$

This suggests the expansion

$$z(\xi,t) = z_0(\xi) + o(1), \tag{61}$$

where z_0 satisfies the differential equation

$$-\frac{\kappa}{2}\beta^{\beta\gamma-1}\left[1+\frac{z_0^2}{4p\beta}\right]^{1-\beta\gamma}z_0=\beta^{\beta\gamma}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma}\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}-\frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma-1}z_0\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2, \tag{62}$$

subject to the boundary conditions (33). To solve (62), note that after simplification it gives

$$\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}=\frac{\gamma}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-1}z_0\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2-\frac{\kappa}{2\beta}\left[1+\frac{z_0^2}{4p\beta}\right]z_0. \tag{63}$$

This equation is similar to (43). The same solution procedure yields

$$\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2=\left[-\frac{\kappa p}{(1-\beta\gamma)}\left(1+\frac{z_0^2}{4p\beta}\right)^{2(1-\beta\gamma)}+C\right]\left(1+\frac{z_0^2}{4p\beta}\right)^{2\beta\gamma},\quad \beta\gamma\neq 1. \tag{64}$$

For sufficiently small κ, ODE (64) is written as

$$\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2=[\mathrm{O}(\kappa)+C]\left(1+\frac{z_0^2}{4p\beta}\right)^{2\beta\gamma}$$

or

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0}=[\mathrm{O}(\kappa)+C]^{-1}\left(1+\frac{z_0^2}{4p\beta}\right)^{-\beta\gamma},$$

which is essentially Eq. (37). Thus, we have the general result (39), and for $\beta\gamma=3/2$ we have (41) and (42).

For large κ and $\beta\gamma>1$, the ODE (64) with boundary conditions (36) and (35) does have a solution, and thus MMPDE5 works satisfactorily in this case, while for large κ and $\beta\gamma<1$, it works unsatisfactorily. The reason is essentially the same as that for MMPDE5 with $\beta\gamma=1$, large constant τ (see Section 4.1, Case 2).

5.2. MMPDE6

The choice for τ for MMPDE6 is the same as that for MMPDE5, i.e., (59). We do not consider the case $\beta\gamma=1$ since it is the case with constant τ which has been discussed in Section 4.2. However, the function z_0 in the expansion of the coordinate transformation (61) now satisfies a different differential equation,

$$\frac{\kappa}{2}\beta^{\beta\gamma-1}\left[1+\frac{z_0^2}{4p\beta}\right]^{1-\beta\gamma}\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}=\beta^{\beta\gamma}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma}\frac{\mathrm{d}^2z_0}{\mathrm{d}\xi^2}-\frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma-1}z_0\left(\frac{\mathrm{d}z_0}{\mathrm{d}\xi}\right)^2, \tag{65}$$

with boundary conditions (33). Letting

$$v=\frac{\mathrm{d}z_0}{\mathrm{d}\xi},$$

we have

$$\frac{\mathrm{d}v}{v}=\beta\gamma\left[1+\frac{z_0^2}{4p\beta}\right]^{-1}\left[1-\frac{\kappa}{2\beta}\left(1+\frac{z_0^2}{4p\beta}\right)\right]^{-1}\mathrm{d}\left(1+\frac{z_0^2}{4p\beta}\right).$$

Its solution can be found as

$$v=C\left|\frac{1+\frac{z_0^2}{4p\beta}}{1-\frac{\kappa}{2\beta}\left(1+\frac{z_0^2}{4p\beta}\right)}\right|^{\beta\gamma}$$

or

$$\frac{\mathrm{d}\xi}{\mathrm{d}z_0}=C^{-1}\left|\left(1+\frac{z_0^2}{2p\beta}\right)^{-1}-\frac{\kappa}{2\beta}\right|^{\beta\gamma}. \tag{66}$$

If $\kappa > 0$ is sufficiently small, then ODE (66) becomes

$$\frac{d\xi}{dz_0} = C^{-1}\left[\left(1+\frac{z_0^2}{2p\beta}\right)^{-1} - O(\kappa)\right]^{\beta\gamma},$$

which is essentially the form Eq. (37). In this case, MMPDE6 works satisfactorily with the general blowup profile (39), and the special forms (41) and (42) when $\beta\gamma = 3/2$.

For sufficiently large κ, Eq. (66) is written as

$$\frac{d\xi}{dz_0} = C^{-1}\left[\frac{\kappa}{2\beta} - \left(1+\frac{z_0^2}{2p\beta}\right)^{-1}\right]^{\beta\gamma}. \tag{67}$$

Integrating (67) in z_0 from z_0^L to z_0^R and applying the boundary conditions (33) gives

$$C = \frac{1}{\xi^R - \xi^L}\int_{z_0^L}^{z_0^R}\left[\frac{\kappa}{2\beta} - \left(1+\frac{z_0^2}{2p\beta}\right)^{-1}\right]^{\beta\gamma} dz_0.$$

Using the approximation boundary conditions (36) and (35), we have $C = +\infty$. Therefore, ODE (67) with boundary conditions (36) and (35) does not have solutions. In other words, few points are distributed in the blowup peak region, and thus the MMPDE fails to work in this case.

5.3. MMPDE4

The dimension Eq. (21) suggests that τ be chosen as

$$\tau = \kappa u^{-\frac{1}{\beta}}. \tag{68}$$

With this choice the MMPDE4 can be reduced to

$$\begin{aligned}
&-\kappa\beta^{\beta\gamma-1}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma+1}\left[\frac{1}{2}[-1+[\alpha-\log(T-t)]^{-1}]z_{\xi\xi} + (T-t)\dot{z}_{\xi\xi}\right] \\
&\quad + \frac{\kappa\gamma\beta^{\beta\gamma-1}}{2p}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma} zz_\xi\left[\frac{1}{2}[-1+[\alpha-\log(T-t)]^{-1}]z_\xi + (T-t)\dot{z}_\xi\right] \\
&\quad = \beta^{\beta\gamma}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma} z_{\xi\xi} - \frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z^2}{4p\beta}\right]^{-\beta\gamma-1} z(z_\xi)^2 + o(1),
\end{aligned} \tag{69}$$

Thus, for any choice of M in the form (12) we can expand $z(\xi,t)$ into (61) with $z_0(\xi)$ satisfying

$$\begin{aligned}
&+\frac{\kappa\beta^{\beta\gamma-1}}{2}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma+1}\frac{d^2z_0}{d\xi^2} - \frac{\kappa\gamma\beta^{\beta\gamma-1}}{4p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma} z_0\left(\frac{dz_0}{d\xi}\right)^2 \\
&\quad = \beta^{\beta\gamma}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma}\frac{d^2z_0}{d\xi^2} - \frac{\gamma\beta^{\beta\gamma}}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma-1} z_0\left(\frac{dz_0}{d\xi}\right)^2,
\end{aligned} \tag{70}$$

subject to the boundary conditions (33). This simplifies to

$$\frac{d^2z_0}{d\xi^2} = \frac{\gamma}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-1} z_0\left(\frac{dz_0}{d\xi}\right)^2.$$

Note that this equation is independent of κ. The transformation

$$v = \frac{dv}{dz_0}$$

leads to

$$\frac{dv}{v} = \frac{\gamma}{2p}\left[1+\frac{z_0^2}{4p\beta}\right]^{-1} z_0\, dz_0,$$

whose solution is

$$v = C\left[1+\frac{z_0^2}{4p\beta}\right]^{\beta\gamma}.$$

Hence

$$\frac{d\xi}{dz_0} = C^{-1}\left[1+\frac{z_0^2}{4p\beta}\right]^{-\beta\gamma},$$

which is exactly the Eq. (37). Thus, the coordinate transformation has the form

$$z(\xi,t) = z_0(\xi) + o(1), \tag{71}$$

where z_0 is given in (38). The general blowup representation Eq. (39) holds true, and in particular, (42) for $\beta\gamma = 3/2$.

The results obtained in this section are summarized in Table 3. It is interesting to note that the MMPDEs are dimensionally homogeneous (or scaling invariant) for all cases and have the dominance of equidistribution when κ is sufficiently small. They work satisfactorily when they have the dominance of equidistribution and in some cases when they do not. This implies that the dominance of equidistribution is sufficient but not necessary.

5.4. Numerical examples

Numerical results are shown in Table 4 and Figs. 6–10. Once again, they are consistent with the theoretical predictions summarized in Table 3. In particular, MMPDEs work satisfactorily when they have the dominance of equidistribution.

Table 3
Summary of results for variable τ. The parameter τ is chosen such that MMPDEs are scaling invariant

MMPDE	$M = u^\gamma$	$\tau > 0$	$\kappa > 0$	Theory	Tests
MMPDE5	$\beta\gamma > 1$	$\kappa u^{\gamma-1/\beta}$	Any	(39), (41) and (42)	Fig. 6
	$\beta\gamma < 1$	$\kappa u^{\gamma-1/\beta}$	Small	(39), (41) and (42)	Not shown
			Large	Unsatisfactory	Fig. 7
MMPDE6	$\beta\gamma \neq 1$	$\kappa u^{\gamma-1/\beta}$	Small	(39), (41) and (42)	Not shown
			Large	Unsatisfactory	Figs. 8 and 9
MMPDE4	$\beta\gamma > 0$	$\kappa u^{-1/\beta}$	Any	(39), (41) and (42)	Fig. 10

They all have the dominance of equidistribution when κ is sufficiently small.

Table 4
Summary of numerical results for variable τ

MMPDEs	κ	$M = u^\gamma$	p	Figs.	Satisfactory	EP dom.	Scaling Inv.
MMPDE5	10^{-5}	$\beta\gamma = 1.5$	3	6	Yes	Yes	Yes
	1	$\beta\gamma = 1.5$	3	Not shown	Yes	No	Yes
	10^{-5}	$\beta\gamma = 2/3$	3	Not shown	Yes	Yes	Yes
	1	$\beta\gamma = 2/3$	3	7	No	No	Yes
MMPDE6	10^{-5}	$\beta\gamma = 1.5$	3	Not shown	Yes	Yes	Yes
	1	$\beta\gamma = 1.5$	3	8	No	No	Yes
	10^{-5}	$\beta\gamma = 2/3$	3	Not shown	Yes	Yes	Yes
	1	$\beta\gamma = 2/3$	3	9	No	No	Yes
MMPDE4	10^{-5}	$\beta\gamma = 1.5$	3	Not shown	Yes	Yes	Yes
	1	$\beta\gamma = 1.5$	3	Not shown	Yes	No	Yes
	10^{-5}	$\beta\gamma = 2/3$	3	10	Yes	Yes	Yes

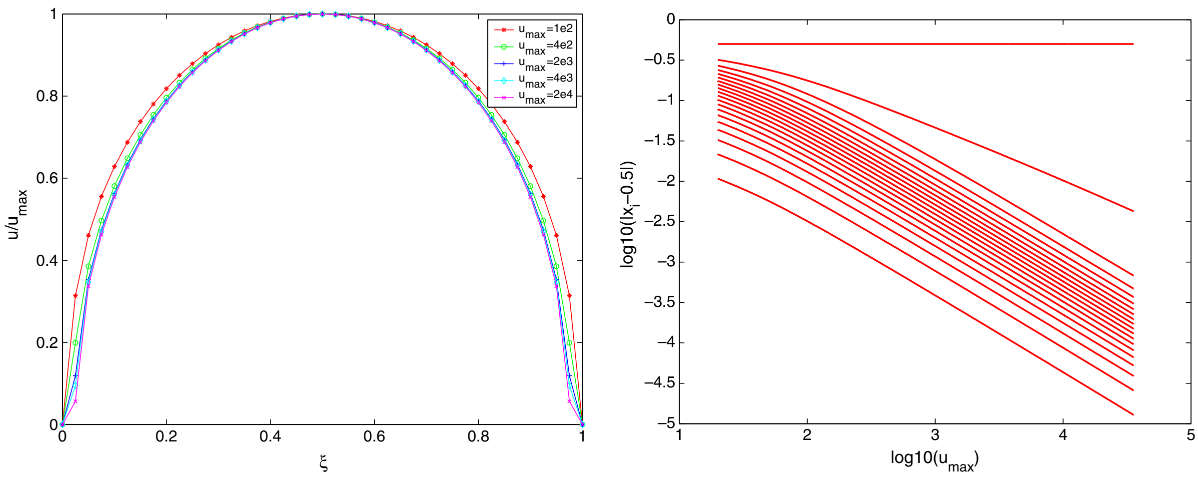

Fig. 6. MMPDE5, $M = u^{1.5(p-1)}$, $p = 3$, $\kappa = 10^{-5}$, $\beta\gamma = 1.5$.

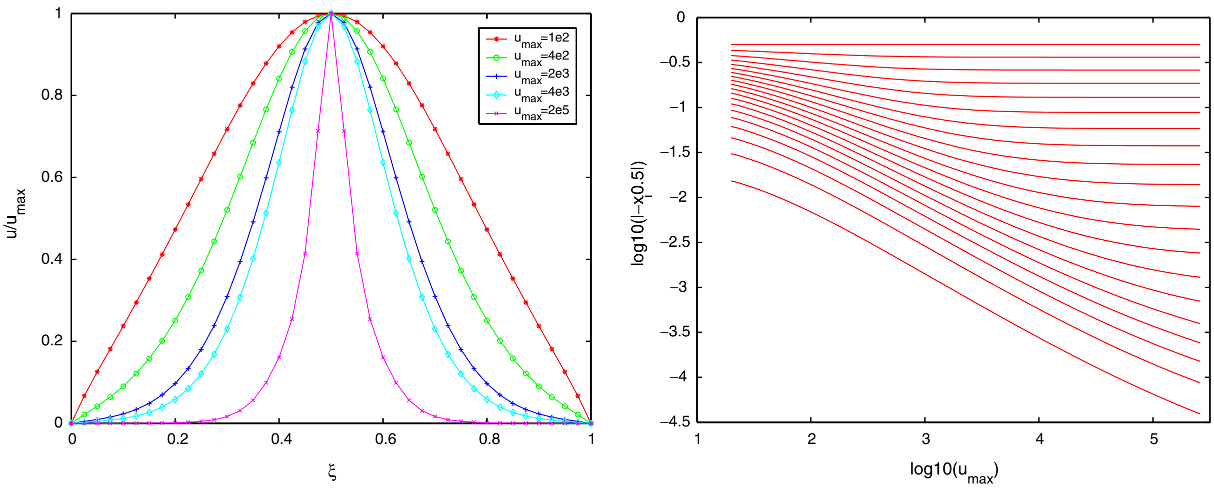

Fig. 7. MMPDE5, $M = u^{2(p-1)/3}$, $p = 3$, $\kappa = 1$, $\beta\gamma = 2/3$.

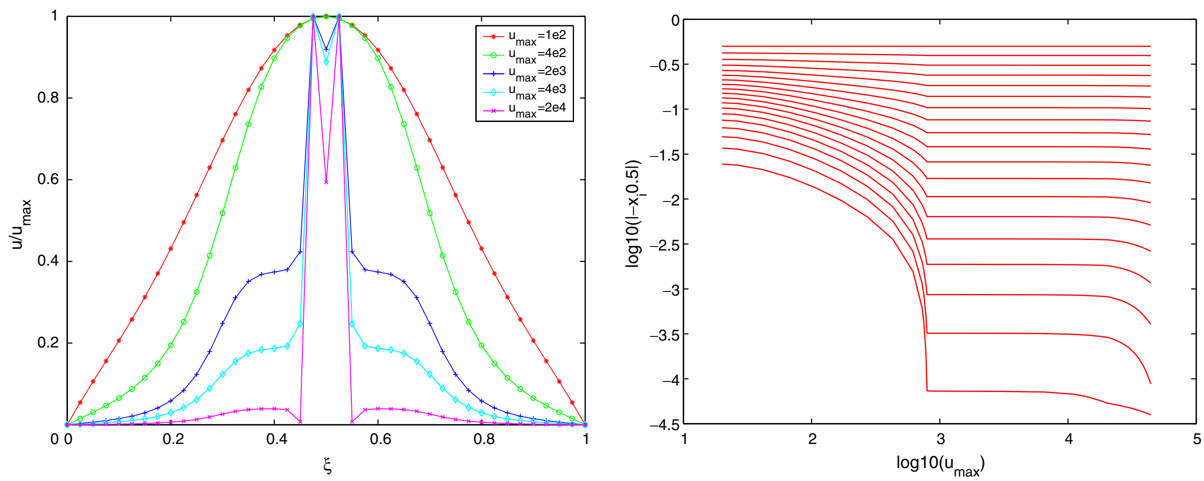

Fig. 8. MMPDE6, $M = u^{1.5(p-1)}$, $p = 3$, $\kappa = 1$, $\beta\gamma = 1.5$.

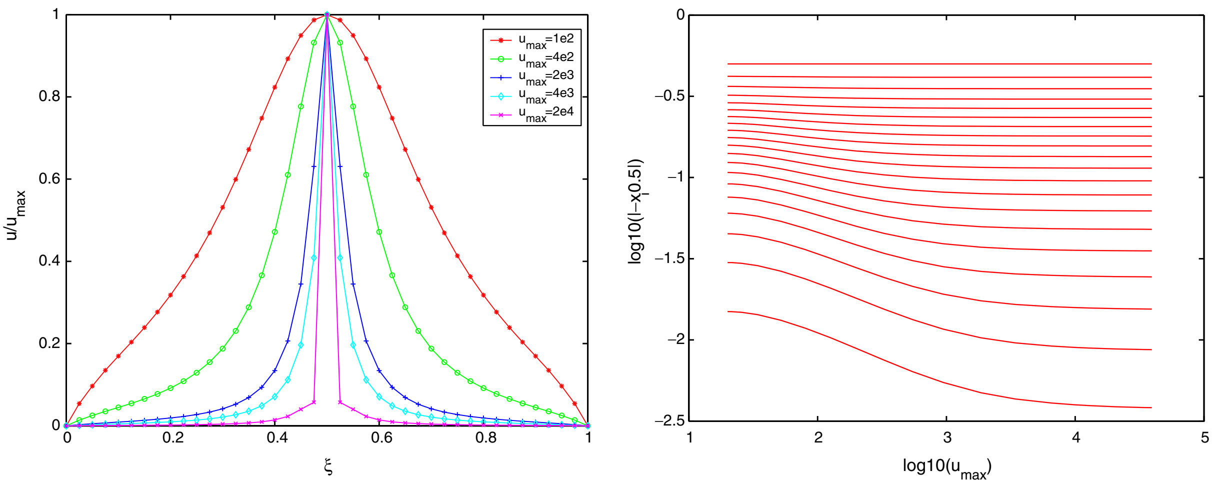

Fig. 9. MMPDE6, $M = u^{2(p-1)/3}$, $p = 3$, $\kappa = 1$, $\beta\gamma = 2/3$.

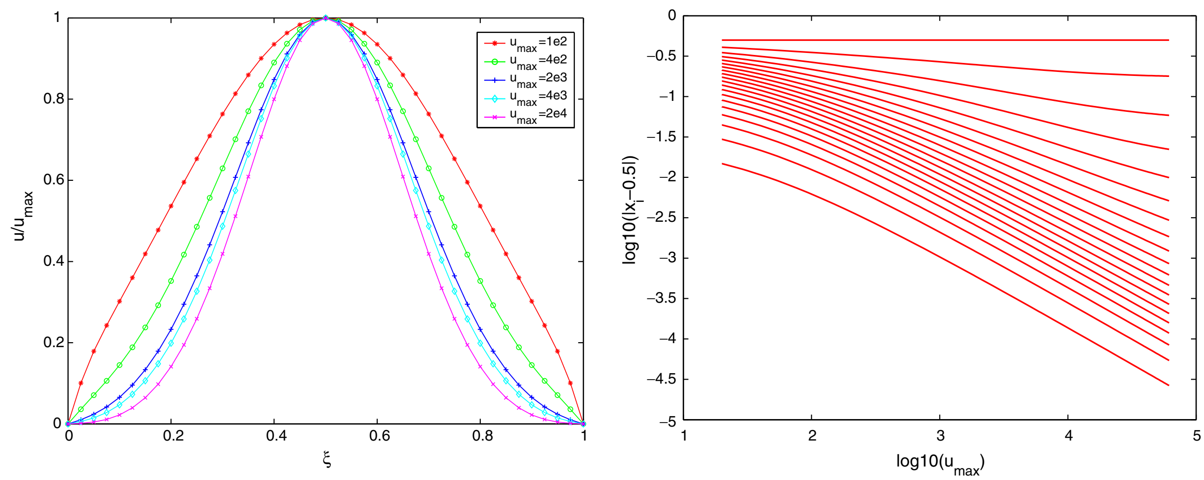

Fig. 10. MMPDE4, $M = u^{2(p-1)/3}$, $p = 3$, $\kappa = 10^{-5}$, $\beta\gamma = 2/3$.

6. Conclusions and comments

In the previous sections we have studied the MMPDE moving mesh method for the numerical solution of blowup problems for the reaction diffusion equation (1). The concept of the dominance of equidistribution is introduced. It represents the fact that the term associated with the equidistribution principle dominates the other terms in a moving mesh equation. We have shown both theoretically and numerically that a moving mesh PDE works satisfactorily when it has the dominance of equidistribution; see Tables 1–4. In addition, the property can be verified straightforwardly using dimensional analysis.

The results obtained in this paper can be regarded as generalizations of the previous work [8] because one example of making an MMPDE have the dominance of equidistribution is to have it preserve the scaling invariance of the underly physical PDE when choosing a small, constant value for τ. As shown in Table 1, the dominance of equidistribution is sufficient for an MMPDE to work satisfactorily whereas the preservation of the scaling invariance is neither sufficient nor necessary in general. In addition, our analysis treats both the case where τ is taken as constant and where it is a function depending on the solutions.

While the study presented here focuses on the classic blowup problem (1), the procedure and the concept of the dominance of equidistribution are very general and can straightforwardly be applied to other types of PDEs with blowup solutions.

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